OPERATIONAL METHODS AND NEUMANN SERIES

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§1

THE operational method furnishes a very simple means of obtaining expansions in Neumann series of given functions. Some examples of the method were given recently by D. P. Banerjee.[1] It is found that many known examples of Neumann series result very simply from known operational forms. Various examples of this are set forth in §§3, 4 below, while some new expansions are obtained in §5. The method is applied in §6 to the Neumann series considered by Kapteyn (Watson, *Bessel Functions*, p. 531; this work will hereafter be referred to as BF.). It is found that the process leads to a formula for the sum different from that given by Kapteyn.

The integral for the sum so obtained can be evaluated explicitly by known formulæ and thus we are able to give an explicit expression, not involving any integrals, for the sum of the Neumann series considered by Kapteyn. Incidentally an entirely independent proof of Kapteyn's formula for the sum is also given (§6).

§2

Suppose that a function $f(z)$ admits the Neumann expansion:

$$f(z) = \sum_{n=0}^{\infty} a_n \, J_{\nu+n}(z) \tag{1}$$

writing $\phi(p)$ for the operational form of $f(z)$, and recalling that the operational form of $J_\nu(z)$ is $p/\sqrt{p^2+1}\,\{p+\sqrt{p^2+1}\}^{-\nu}$. $R(\nu) > -1$, we have from (1)

$$\phi(p) = \frac{p}{\sqrt{p^2+1}} \sum_{n=0}^{\infty} a_n \{p+\sqrt{p^2+1}\}^{-\nu-n} \tag{2}$$

If we now write

$$\zeta = \sqrt{p^2+1} - p, \quad 1/\zeta = \sqrt{p^2+1} + p \tag{3}$$

this gives

$$\frac{1+\zeta^2}{1-\zeta^2}\,\phi\left(\frac{1-\zeta^2}{2\zeta}\right) = \sum_{n=0}^{\infty} a_n \, \zeta^{\nu+n} \tag{4}$$

Conversely if we know $\phi(p)$ and the expansion (4), we may derive (1) by interpreting (4) term-by-term. On the other hand if (1) be regarded as known, then (4) furnishes the operational form $\phi(p)$ of $f(z)$; this may be useful in some cases to obtain an operational form not otherwise known.

§3

We shall now consider applications of the above method.

(i) $\phi(p) = p^{-\nu}, f(z) = \dfrac{z^\nu}{\Gamma(\nu+1)}$, $R(\nu) > -1$.

$$\frac{1+\zeta^2}{1-\zeta^2}\phi\left(\frac{1-\zeta^2}{2\zeta}\right) = \frac{(1+\zeta^2)\,(2\zeta)^\nu}{(1-\zeta^2)^{\nu+1}} = 2^\nu \sum_{n=0}^{\infty} \frac{\Gamma(\nu+n)}{n!\,\Gamma(\nu+1)}\,(\nu+2n)\,\zeta^{\nu+2n}$$

Hence by (4) and (1)

$$\frac{z^\nu}{\Gamma(\nu+1)} = 2^\nu \sum_{n=0}^{\infty} \frac{\Gamma(\nu+n)}{n!\,\Gamma(\nu+1)}\,(\nu+2n)\,J_{\nu+2n}(z)$$

or

$$(\tfrac{1}{2}z)^\nu = \sum_{n=0}^{\infty} \frac{(\nu+2n)\,\Gamma(\nu+n)}{n!}\,J_{\nu+2n}(z) \tag{5}$$

which is the Schlomilch-Gegenbauer expansion of $(\frac{1}{2}z)^\nu$ (BF. 138).

(ii) $\phi(p) = \dfrac{p}{(p^2+1)^{\nu+\frac{1}{2}}}$, $f(z) = \sqrt{\pi}.\ \dfrac{z^\nu J_\nu(z)}{2^\nu\,\Gamma(\nu+\frac{1}{2})}$

$$\frac{1+\zeta^2}{1-\zeta^2}\phi\left(\frac{1-\zeta^2}{2\zeta}\right) = \frac{1+\zeta^2}{1-\zeta^2}\,\frac{2^{2\nu}\,\zeta^{2\nu}(1-\zeta^2)}{(1+\zeta^2)^{2\nu+1}} = \frac{(2\zeta)^{2\nu}}{(1+\zeta^2)^{2\nu}}$$

$$= 2^{2\nu}\,\zeta^{2\nu} \sum_{n=1}^{\infty} (-1)^n \frac{\Gamma(2\nu+n)}{n!\,\Gamma(2\nu)}\,\zeta^{2n}$$

whence

$$\sqrt{\pi}\,\frac{z^\nu J_\nu(z)}{2^\nu\,\Gamma(\nu+\frac{1}{2})} = 2^{2\nu} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(2\nu+n)}{n!\,\Gamma(2\nu)}\,J_{2\nu+2n}(z) \tag{6}$$

which is the particular case $\mu = 2\nu$ of the formula

$$(\tfrac{1}{2}z)^{\mu-2\nu}\,J_\nu(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\mu+n)\,\Gamma(\nu+1-\mu)}{n!\,\Gamma(\nu+1-\mu-n)\,\Gamma(\nu+n+1)}\,(\mu+2n)\,J_{\mu+2n}(z) \tag{7}$$

(BF. 139).

(iii) $\phi(p) = 1/\sqrt{p^2+1}\,\{p+\sqrt{p^2+1}\}^\nu$, $f(z) = \displaystyle\int_0^z J_\nu(t)\,dt$.

$$\frac{1+\zeta^2}{1-\zeta^2}\phi\left(\frac{1-\zeta^2}{2\zeta}\right) = \frac{2\,\zeta^{\nu+1}}{1-\zeta^2} = \sum_{n=0}^{\infty} \zeta^{\nu+2n+1}$$

so that

$$\int_0^z J_\nu(t)\,dt = 2\sum_{n=0}^{\infty} J_{\nu+2n+1}(z) \tag{8}$$

Similarly taking $\phi(p) = \{p + \sqrt{p^2+1}\}^{-\nu}$, $f(z) = \nu \int_0^z \frac{J_\nu(t)}{t}\,dt$, $R(\nu) > 0$

we get $$\nu \int_0^z \frac{J_\nu(t)t}{t}\,dt = J_\nu(z) + 2\sum_{n=1}^{\infty} J_{\nu+2n}(z) \tag{9}$$

(BF. 530).

(iv) Next consider $f(z) = z^\nu e^{iz\cos\phi}$, $R(\nu) > -1$.

Here $$\phi(p) = \frac{p\,\Gamma(\nu+1)}{(p - i\cos\phi)^{\nu+1}}$$

$$\therefore \quad \frac{1+\zeta^2}{1-\zeta^2}\,\phi\left(\frac{1-\zeta^2}{2\zeta}\right) = \Gamma(\nu+1)\,\frac{(2\zeta)^\nu(1+\zeta^2)}{(1-2i\zeta\cos\phi-\zeta^2)^{\nu+1}}.$$

Now it is easily proved that

$$\frac{\nu(1-\zeta^2)}{(1-2\zeta\cos\phi+\zeta^2)^{\nu+1}} = \sum_{n=0}^{\infty}(n+\nu)\,C^\nu_n(\cos\phi)\,\zeta^n$$

where $C^\nu_n(\cos\phi)$ is the coefficient of h^n in the expansion of $(1-2h\cos\phi+h^2)^{-\nu}$. Changing ζ into $i\zeta$ we get

$$\frac{\nu(1+\zeta^2)}{(1-2i\zeta\cos\phi-\zeta^2)^{\nu+1}} = \sum_{n=0}^{\infty}(n+\nu)\,C^\nu_n(\cos\phi)\,i^n\zeta^n \tag{9 a}$$

Thus

$$\frac{1+\zeta^2}{1-\zeta^2}\,\phi\left(\frac{1-\zeta^2}{2\zeta}\right) = 2^\nu\,\Gamma(\nu)\sum_{n=0}^{\infty}(n+\nu)\,i^n\zeta^{\nu+n}\,C^\nu_n(\cos\phi)$$

Hence

$$z^\nu e^{iz\cos\phi} = 2^\nu\,\Gamma(\nu)\sum_{n=0}^{\infty}(\nu+n)\,i^n J_{\nu+n}(z)\,C^\nu_n(\cos\phi) \tag{10}$$

which is the well-known expansion of Bauer and Gegenbauer (BF. 368).

§4

From Weber's first and second exponential integrals (BF. 394–5) we have

$$\frac{z^{\nu/2} J_\nu(\rho\sqrt{z})}{\rho^\nu} \supset \frac{1}{(2p)^\nu}\,e^{-\rho^2/4p} \tag{11}^*$$

* The notation $f(z) \supset \phi(p)$ implies that $\phi(p)$ is the operational form of $f(z)$.

and

$$J_\nu(a\sqrt{z})\, I_\nu(b\sqrt{z}) \supset e^{-\frac{a^2-b^2}{4p}} J_\nu\left(\frac{ab}{2p}\right) \tag{12}$$

From (11) we have, on taking $\rho^2 = a^2 - b^2 - 2\,iab\cos\phi$,

$$\frac{z^{\nu/2} J_\nu\left(\sqrt{z(a^2-b^2-2\,iab\cos\phi)}\right)}{(a^2-b^2-2\,iab\cos\phi)^{\nu/2}} \supset \frac{1}{(2p)^\nu}\, exp\left\{-\frac{a^2-b^2-2\,iab\cos\phi}{4p}\right\}$$

$$= \left(\frac{1}{ab}\right)^\nu e^{-\frac{a^2-b^2}{4p}} \left(\frac{ab}{2p}\right)^\nu e^{i\left(\frac{ab}{2p}\right)\cos\phi}$$

$$= \left(\frac{1}{ab}\right)^\nu .\ e^{-\frac{a^2-b^2}{4p}} .\ 2^\nu \Gamma(\nu) \sum_{m=0}^{\infty} (\nu+m)\, i^m C^\nu_m(\cos\phi)\, J_{\nu+m}\left(\frac{ab}{2p}\right)$$

by (10),

$$= \left(\frac{2}{ab}\right)^\nu \Gamma(\nu) \sum_{m=0}^{\infty} (\nu+m)\ i^m C^\nu_m(\cos\phi).\ e^{-\frac{a^2-b^2}{4p}} J_{\nu+m}\left(\frac{ab}{2p}\right).$$

Interpreting the right hand side term-by-term by means of (12), we get

$$\frac{z^{\nu/2} J_\nu\left(\sqrt{z(a^2-b^2-2\,iab\cos\phi)}\right)}{(a^2-b^2-2\,iab\cos\phi)^{\nu/2}} =$$

$$\left(\frac{2}{ab}\right)^\nu \Gamma(\nu) \sum_{m=0}^{\infty} (\nu+m)\, i^m C^\nu_m(\cos\phi) J_{\nu+m}(a\sqrt{z})\, I_{\nu+m}(b\sqrt{z})$$

Changing b into $-ib$ and writing $t = a\sqrt{z}$, $u = b\sqrt{z}$, we get finally

$$\frac{J_\nu\left(\sqrt{t^2+u^2-2\,ut\cos\phi}\right)}{(t^2+u^2-2\,ut\cos\phi)^{\nu/2}} = 2^\nu \Gamma(\nu) \sum_{m=0}^{\infty} (\nu+m) \frac{J_{\nu+m}(t)}{t^\nu} \cdot \frac{J_{\nu+m}(u)}{u^\nu} C^\nu_m(\cos\phi) \tag{13}$$

which is Gegenbauer's addition-theorem (BF. 363).

We shall next consider $f(z) = \left(\frac{z-b}{z+b}\right)^{\nu/2} J_\nu(\sqrt{z^2-b^2})$. It is known[2] that the operational form of $f(z)$ is given by

$$\phi(p) = \frac{p}{\sqrt{p^2+1}}\{\sqrt{p^2+1}+p\}^{-\nu}.\ e^{-b\sqrt{p^2+1}}$$

By a theorem in the Operational Calculus[3] we know that if $\phi(p)$ is the operational form of $f(z)$, then $e^{-ap}\phi(p)$ is the operational form of $f(z-a)$. Applying this theorem, after taking $b = -iu\sin\phi$, $a = u\cos\phi$, we find that

$$f(z) = \left(\frac{z-u\cos\phi+iu\sin\phi}{z-u\cos\phi-iu\sin\phi}\right)^{\nu/2} J_\nu\left(\sqrt{z^2+u^2-2\,uz\cos\phi}\right)$$

$$\supset \frac{p}{\sqrt{p^2+1}}\{\sqrt{p^2+1}+p\}^{-\nu}\, e^{-pu\cos\phi + iu\sin\phi\sqrt{p^2+1}}$$

$$i.e.\ \left(\frac{z-ue^{-i\phi}}{z-ue^{i\phi}}\right)^{\nu/2} J_\nu\left(\sqrt{z^2+u^2-2\,zu\cos\phi}\right)$$

$$\supset \frac{p}{\sqrt{p^2+1}}\{\sqrt{p^2+1}+p\}^{-\nu}\exp\left\{\tfrac{1}{2}ue^{i\phi}(\sqrt{p^2+1}-p)-\tfrac{1}{2}ue^{-i\phi}(\sqrt{p^2+1}+p)\right\}$$
$$=\phi(p).$$

$$\therefore \left(\frac{1+\zeta^2}{1-\zeta^2}\right)\phi\left(\frac{1-\zeta^2}{2\,\zeta}\right)=\zeta^\nu e^{\frac{1}{2}u\left(\zeta e^{i\phi}-\frac{1}{\zeta e^{i\phi}}\right)}$$

$$=\zeta^\nu\sum_{m=-\infty}^{+\infty}\zeta^m e^{im\phi}J_m(u)$$

Hence

$$\left(\frac{z-ue^{-i\phi}}{z-ue^{i\phi}}\right)^{\nu/2} J_\nu\left(\sqrt{z^2+u^2-2\,zu\cos\phi}\right)=\sum_{m=-\infty}^{+\infty}J_{\nu+m}(z)\,J_m(u)\,e^{mi\phi} \quad (14)$$

which is Graf's generalisation of Neumann's addition formula (BF. 359).

§5

As a further example of the above method, let us take $f(z)=\mathbf{H}_\nu(z)$, the Struve's function of order ν. If $R(\nu)>-\frac{1}{2}$, we have (BF. 328).

$$\mathbf{H}_\nu(z)=\frac{2\,(\frac{1}{2}z)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}\int_0^{\frac{\pi}{2}}\sin(z\cos\phi)\,\sin^{2\nu}\phi\,d\phi$$

The operational form is

$$\phi(p)=p\int_0^\infty e^{-pz}\,\mathbf{H}_\nu(z)\,dz$$

$$=\frac{p}{2^{\nu-1}\,\Gamma(\nu+\frac{1}{2})\,\Gamma(\frac{1}{2})}\int_0^{\frac{\pi}{2}}\sin^{2\nu}\phi\,d\phi\int_0^\infty e^{-pz}\sin(z\cos\phi)\,z^\nu\,dz$$

i.e. $\phi(p)$ is the imaginary part of

$$\psi(p)=\frac{p}{2^{\nu-1}\,\Gamma(\nu+\frac{1}{2})\,\Gamma(\frac{1}{2})}\int_0^{\frac{\pi}{2}}\sin^{2\nu}\phi\,d\phi\int_0^\infty e^{-(pz-iz\cos\phi)}\,z^\nu\,dz$$

$$=\frac{p\,\Gamma(\nu+1)}{2^{\nu-1}\,\Gamma(\nu+\frac{1}{2})\,\Gamma(\frac{1}{2})}\int_0^{\frac{\pi}{2}}\frac{\sin^{2\nu}\phi}{(p-i\cos\phi)^{\nu+1}}\,d\phi$$

We have then

$$\frac{1+\zeta^2}{1-\zeta^2}\psi\left(\frac{1-\zeta^2}{2\zeta}\right) = \frac{\Gamma(\nu+1)}{2^{\nu-1}\,\Gamma(\nu+\frac{1}{2})\,\Gamma(\frac{1}{2})}\cdot\frac{1+\zeta^2}{1-\zeta^2}\cdot\frac{1-\zeta^2}{2\zeta}\cdot(2\zeta)^{\nu+1}$$

$$\times\int_0^{\frac{\pi}{2}}\frac{\sin^{2\nu}\phi}{(1-\zeta^2-2i\,\zeta\cos\phi)^{\nu+1}}\,d\phi$$

$$=\frac{\Gamma(\nu)}{2^{\nu-1}\,\Gamma(\nu+\frac{1}{2})\,\Gamma(\frac{1}{2})}(2\zeta)^{\nu}\cdot\sum_{n=0}^{\infty}(n+\nu)\,i^n\,\zeta^n\int_0^{\frac{\pi}{2}}C^{\nu}{}_n(\cos\phi)\sin^{2\nu}\phi\,d\phi \quad (15)$$

on simplifying and using (9 a).

In order to calculate the definite integral we use the formula[4]

$$C_n^{\nu}(\cos\phi) = \frac{(-2)^n\,\Gamma(\nu+n)\,\Gamma(n+2\nu)}{n\,!\,\Gamma(\nu)\,\Gamma(2n+2\nu)}\cdot\frac{1}{\sin^{2\nu-1}\phi}\cdot\frac{d^n}{dx^n}\{(1-x^2)^{n+\nu-\frac{1}{2}}\}$$

where $x=\cos\phi$. We find

$$\int_0^{\frac{\pi}{2}} C_n^{\nu}(\cos\phi)\sin^{2\nu}\phi\,d\phi$$

$$=\frac{(-2)^n\,\Gamma(\nu+n)\,\Gamma(n+2\nu)}{n\,!\,\Gamma(\nu)\,\Gamma(2n+2\nu)}\int_0^1\frac{d^n}{dx^n}\{(1-x^2)^{n+\nu-\frac{1}{2}}\}\,dx$$

$$=\frac{(-2)^n\,\Gamma(\nu+n)\,\Gamma(n+2\nu)}{n\,!\,\Gamma(\nu)\,\Gamma(2n+2\nu)}\cdot\frac{(n-1)\,!\,\Gamma(\nu+1)\,\Gamma(2n+2\nu)}{(-2)^{n-1}\,\Gamma(\nu+n)\,\Gamma(n+2\nu+1)}$$

$$\times\left[\sin^{2\nu}\phi.\ \ C_{n-1}^{\nu+1}(\cos\phi)\right]^0_{\pi/2}$$

$$=\frac{-2\nu}{n\ (n+2\nu)}\,C_{n-1}^{\nu+1}(0)$$

Now from the expansion $(1+h^2)^{-\nu}=\Sigma h^n\,C_n^{\nu}(0)$, it is readily seen that

$$C^{\nu}{}_{2r-1}(0)=0,\quad C^{\nu}{}_{2r}(0)=\frac{(-1)^r\,\Gamma(\nu+r)}{r\,!\,\Gamma(\nu)}$$

Thus we find

$$\left.\begin{aligned}&\int_0^{\frac{\pi}{2}} C^{\nu}{}_{2r}(\cos\phi)\sin^{2\nu}\phi\,d\phi = 0\\ &\int_0^{\frac{\pi}{2}} C^{\nu}{}_{2r+1}(\cos\phi)\sin^{2\nu}\phi\,d\phi = \frac{2\nu}{(2r+1)(2r+2\nu+1)}(-1)^r.\frac{\Gamma(\nu+r+1)}{r!\,\Gamma(\nu+1)}.\end{aligned}\right\}\quad(16)$$

Hence we get from (15)

$$\frac{1+\zeta^2}{1-\zeta^2}\psi\left(\frac{1-\zeta^2}{2\zeta}\right)$$

$$=\frac{2\nu\,\Gamma(\nu)\,(2\zeta)^\nu}{2^{\nu-1}\,\Gamma(\nu+\frac{1}{2})\,\Gamma(\frac{1}{2})}\sum_{r=0}^{\infty}\frac{(2r+\nu+1)\,i^{2r+1}\,\zeta^{2r+1}}{(2r+1)(2r+2\nu+1)}\cdot\frac{(-1)^r.\,\Gamma(\nu+r+1)}{r\,!\,\Gamma(\nu+1)}$$

$$=\frac{4\,i\zeta^\nu}{\Gamma(\nu+\frac{1}{2})\,\Gamma(\frac{1}{2})}\sum_{r=0}^{\infty}\frac{(2\,r+\nu+1)\;\Gamma(\nu+r+1)}{r\,!\,(2\,r+1)\;(2\,r+2\,\nu+1)}\,\zeta^{2r+1}$$

Hence

$$\frac{1+\zeta^2}{1-\zeta^2}\phi\left(\frac{1-\zeta^2}{2\zeta}\right)=\frac{4}{\Gamma(\nu+\frac{1}{2})\;\Gamma(\frac{1}{2})}\sum_{r=0}^{\infty}\frac{(2\,r+\nu+1)\;\Gamma(\nu+r+1)}{r!\;(2\,r+1)\;(2\,r+2\,\nu+1)}\zeta^{\nu+2r+1}$$

and therefore, if $R(\nu)>-\frac{1}{2}$

$$\mathbf{H}_\nu(z)=\frac{4}{\Gamma(\nu+\frac{1}{2})\;\Gamma(\frac{1}{2})}\sum_{r=0}^{\infty}\frac{(2\,r+\nu+1)\;\Gamma(\nu+r+1)}{r!\;(2\,r+1)\;(2\,r+2\,\nu+1)}\,J_{\nu+2r+1}(z)\tag{17}$$

The proof in the above form is valid only for real ν and z, but the truth of the result for general values of ν and z, provided only that $R(\nu)>-\frac{1}{2}$, may be inferred by analytic continuation.

By changing z into iz, we derive

$$\mathbf{L}_\nu(z)=\frac{4}{\Gamma(\nu+\frac{1}{2})\;\Gamma(\frac{1}{2})}\sum_{r=0}^{\infty}(-1)^r\frac{(2\,r+\nu+1)\;\Gamma(\nu+r+1)}{r\,!\,(2\,r+1)\,(2r+2\,\nu+1)}\,I_{\nu+2r+1}(z)\tag{18}$$

where $\mathbf{L}_\nu(z)$ is the modified Struve's function (BF. 329).

A formula analogous to (14) above may be obtained by starting from $f(z)=$

$$\left(\frac{z-b}{z+b}\right)^{\frac{1}{2}\nu}\mathbf{Y}_\nu\,(a\,\sqrt{z^2-b^2})\supset\frac{p}{\sqrt{p^2+1}}\,\frac{e^{-b\sqrt{p^2+1}}}{\sin\nu\pi}\,\{(a/P)^\nu\cos\nu\pi-(P/a)^\nu\}\tag{19}$$

where $P=\sqrt{p^2+1}+p$, and $-1<R(\nu)<+1,\ \nu\neq 0.$[5]

Here

$$\frac{1+\zeta^2}{1-\zeta^2}\phi\left(\frac{1-\zeta^2}{2\zeta}\right)=\frac{1}{\sin\nu\pi}e^{-\frac{b}{2}\left(\zeta+\frac{1}{\zeta}\right)}\{(a\zeta)^\nu\cos\nu\pi-(a\zeta)^{-\nu}\}$$

$$=\frac{1}{\sin\nu\pi}\,\{(a\zeta)^\nu\cos\nu\pi-(a\zeta)^{-\nu}\}\sum_{n=-\infty}^{+\infty}(-1)^n\,\zeta^n\,I_n(b)$$

Hence

$$\left(\frac{z-b}{z+b}\right)^{\frac{1}{2}\nu}\mathbf{Y}_\nu\,(a\,\sqrt{z^2-b^2})=\frac{1}{\sin\nu\pi}\left\{a^\nu\cos\nu\pi\sum_{n=-\infty}^{+\infty}(-1)^n\,J_{n+\nu}(z)\,I_n(b)\right.$$

$$\left.-a^{-\nu}\sum_{n=-\infty}^{+\infty}(-1)^n\,J_{n-\nu}(z)\,I_n(b)\right\}\tag{20}$$

An example of a slightly more complicated kind is furnished by taking[6]

$$f(z) = (1 - z^2)^{\nu - \frac{1}{2}} \supset \sqrt{\pi}\, \Gamma(\nu + \tfrac{1}{2})\, p\, (2/p)^{\nu}\, \mathrm{I}_{\nu}(p), \quad |t| < 1, \ \mathrm{R}(\nu) > -\tfrac{1}{2}.$$

Now

$$(2/p)^{\nu}\, \mathrm{I}_{\nu}(p) = \sum_{m=0}^{\infty} \frac{(p/2)^{2m}}{m!\, \Gamma(\nu + m + 1)}$$

and

$$p^{2m} = (2\zeta)^{-2m}(1 - \zeta^2)^{2m} = (2\zeta)^{-2m} \sum_{r=0}^{2m} (-1)^r \frac{(2m)!}{r!\,(2m - r)!} \zeta^{2r}$$

$$= 2^{-2m}(-1)^m \sum_{s=-m}^{+m} (-1)^s \frac{(2m)!}{(m + s)!\,(m - s)!} \zeta^{2s}$$

$$\therefore (2/p)^{\nu}\, \mathrm{I}_{\nu}(p) = \sum_{m=0}^{\infty} \frac{(-1)^m (2m)!}{2^{4m}\, m!\, \Gamma(\nu + m + 1)} \sum_{s=-m}^{m} (-1)^s \frac{\zeta^{2s}}{(m+s)!\,(m-s)!}$$

$$= \frac{{}_1\mathrm{F}_2(\tfrac{1}{2};\, 1,\, \nu + 1;\, -\tfrac{1}{4})}{\Gamma(\nu + 1)} + \sum_{s=1}^{\infty} \frac{{}_1\mathrm{F}_2(s + \tfrac{1}{2};\, 2s + 1,\, \nu + s + 1;\, -\tfrac{1}{4})}{2^{4s}\, s!\, \Gamma(\nu + s + 1)} (\zeta^{2s} + \zeta^{-2s})$$

$$= \sum_{s=0}^{\infty} \epsilon_s \frac{{}_1\mathrm{F}_2(s + \tfrac{1}{2};\, 2s+1,\, \nu + s + 1;\, -\tfrac{1}{4})}{2^{4s+1}\, s!\, \Gamma(\nu + s + 1)} (\zeta^{2s} + \zeta^{-2s}) \tag{21}$$

where ϵ_s is Neumann's factor (BF. 22).

Now, if $\phi_1(p) \subset f_1(z)$ and $\phi_2(p) \subset f_2(z)$, then[7] $\dfrac{\phi_1(p)\, \phi_2(p)}{p}$ $\subset \displaystyle\int_0^z f_1(z - t)\, f_2(t)\, dt$. Taking $\phi_1(p) = \dfrac{p}{\sqrt{p^2 + 1}} \subset \mathrm{J}_0(z)$, and $\phi_2(p) =$ $\sqrt{\pi}\, \Gamma(\nu + \tfrac{1}{2})\, p\, (2/p)^{\nu}\, \mathrm{I}_{\nu}(p) \subset (1 - z^2)^{\nu - \frac{1}{2}}$, we get $\mathrm{F}(z) = \displaystyle\int_0^z \mathrm{J}_0(z - t)\, (1 - t^2)^{\nu - \frac{1}{2}}\, dt$

$$\supset \frac{1}{p} \cdot \frac{p}{\sqrt{p^2 + 1}} \cdot \sqrt{\pi}\, \Gamma(\nu + \tfrac{1}{2})\, p\, (2/p)^{\nu}\, \mathrm{I}_{\nu}(p)$$

$$= \sqrt{\pi}\, \Gamma(\nu + \tfrac{1}{2}) \cdot \frac{p}{\sqrt{p^2 + 1}} (2/p)^{\nu}\, \mathrm{I}_{\nu}(p) = \phi(p).$$

$$\therefore \frac{1 + \zeta^2}{1 - \zeta^2} \phi\left(\frac{1 - \zeta^2}{2\zeta}\right) = \sqrt{\pi}\, \Gamma(\nu + \tfrac{1}{2}) \cdot (2/p)^{\nu}\, \mathrm{I}_{\nu}(p)$$

$$= \sqrt{\pi}\, \Gamma(\nu + \tfrac{1}{2}) \sum_{s=0}^{\infty} \epsilon_s \frac{{}_1\mathrm{F}_2(s + \tfrac{1}{2};\, 2s + 1,\, \nu + s + 1;\, -\tfrac{1}{4})}{2^{4s+1}\, s!\, \Gamma(\nu + s + 1)} (\zeta^{2s} + \zeta^{-2s})$$

by (21). Hence

$$\int_0^z J_0(z-t)(1-t^2)^{\nu-\frac{1}{2}}\,dt$$

$$= \sqrt{\pi}\,\Gamma(\nu+\tfrac{1}{2}) \sum_{s=0}^{\infty} \epsilon_s \frac{{}_1F_2(s+\frac{1}{2};\,2s+1,\,\nu+s+1;\,-\frac{1}{4})}{2^{4s}\,s\,!\,\Gamma(\nu+s+1)}\,J_{2s}(z) \quad (22)$$

§6

Next consider the Neumann series summed by Kapteyn (BF. 531):

$$f(z,\alpha) = \sum_{n=1}^{\infty} n\,J_n(z)\,J_n(\alpha)$$

writing $z = a\sqrt{t}$, $\alpha = b\sqrt{t}$, this becomes

$$f(z,\alpha) \equiv F(t) = \sum_{n=1}^{\infty} n\,J_n(a\sqrt{t})\,J_n(b\sqrt{t})$$

$$\supset \sum_{n=1}^{\infty} n\,I_n\left(\frac{a\,b}{2\,p}\right) e^{-\frac{a^2+b^2}{4p}}, \text{ by (12)}$$

$$= \frac{ab}{4p} \sum_{n=1}^{\infty} \left\{ I_{n-1}\left(\frac{ab}{2p}\right) - I_{n+1}\left(\frac{ab}{2p}\right) \right\} e^{-\frac{a^2+b^2}{4p}}$$

$$= \frac{ab}{4p}\left[e^{-\frac{a^2+b^2}{4p}} I_0\left(\frac{ab}{2p}\right) + e^{-\frac{a^2+b^2}{4p}} I_1\left(\frac{ab}{2p}\right)\right]$$

$$\subset \frac{ab}{4}\int_0^t [J_0(a\sqrt{u})\,J_0(b\sqrt{u}) + J_1(a\sqrt{u})\,J_1(b\sqrt{u})]\,du$$

on interpreting back by means of (12). If we put $u = tv^2$, the last integral becomes

$$\frac{\alpha z}{2}\int_0^1 [J_0(zv)\,J_0(\alpha v) + J_1(zv)\,J_1(\alpha v)]\,v dv.$$

We thus obtain

$$\sum_{n=1}^{\infty} n\,J_n(z)\,J_n(\alpha) = \frac{\alpha z}{2}\int_0^1 [J_0(\alpha v)\,J_0(zv) + J_1(\alpha v)\,J_1(zv)]\,v dv \quad (23)$$

In Watson's book the sum of the series on the left is given as

$$\frac{z}{2}\int_0^{\alpha} \frac{J_1(z-t)}{z-t}\,J_0(\alpha-t)\,dt.$$

We are thus led to the following identity*

$$\int_0^a \frac{J_1(z-t)}{z-t} J_0(a-t)\,dt = a \int_0^1 [J_0(av) J_0(zv) + J_1(av) J_1(zv)]\, v dv \quad (24)$$

The integral on the right in (24) may be evaluated by known formulæ [BF. 134, eq. (8)] and the result is

$$\sum_{n=1}^{\infty} n J_n(z) J_n(a) = \frac{z}{2} \int_0^a \frac{J_1(z-t)}{z-t} J_0(a-t)\,dt$$

$$= \frac{az\{J_1(a) J_0(z) - J_0(a) J_1(z)\}}{2(a-z)}. \quad (25)$$

The following is an independent proof of Kapteyn's formula: We have from (13), taking $\phi = 0$, and $\nu = 1$

$$\frac{J_1(z-t)}{z-t} = 2^1 \Gamma(1) \sum_{m=0}^{\infty} (1+m) \frac{J_{m+1}(z)}{z} \frac{J_{m+1}(t)}{t} \cdot \frac{\Gamma(m+2)}{m!\,\Gamma(2)}$$

$$= 2 \sum_{m=0}^{\infty} (m+1)^2 \frac{J_{m+1}(z)}{z} \cdot \frac{J_{m+1}(t)}{t}$$

$$\therefore \int_0^a \frac{J_1(z-t)}{z-t} J_0(a-t)\,dt = \frac{2}{z} \sum_{m=0}^{\infty} (m+1)^2 J_{m+1}(z) \int_0^a \frac{J_{m+1}(t) J_0(a-t)}{t}\,dt$$

$$= \frac{2}{z} \sum_{m=0}^{\infty} (m+1) J_{m+1}(z) J_{m+1}(a)$$

on making use of Bateman's integral (BF, 380, eqn (3). Thus we get

$$\sum_{n=1}^{\infty} n J_n(z) J_n(a) = \frac{z}{2} \int_0^a \frac{J_1(z-t)}{z-t} J_0(a-t)\,dt$$

as desired.

REFERENCES

1. D. P. Banerjee .. *Quart. J. Math.* (Oxford), 1939, **10**, 261.
2. McLachlan .. *Phil. Mag.*, 1937, **23**, 918.
3. ——— .. *Complex Variable and Operational Calculus*, Cambridge, 1939, 229.
4. Whittaker and Watson .. *Modern Analysis*, Fourth Edition, p. 329.
5. McLachlan .. *Loc. cit. ref.* 2.
6. ——— .. *Ibid.*
7. Goldstein .. *Proc. Lond. Math. Soc.*, (2), 1932, **34**, 104.

* The truth of (24) may be verified in the following particular cases: $a = 0$, $z = a$ and $z = 0$, using, in the last case, Bateman's integral to evaluate the integral on the left. But I have not been able to obtain a general proof of the identity.